

# Switched Systems With Multiple Invariant Sets<sup>☆</sup>

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## Abstract

This paper explores dwell time constraints on switched systems with multiple, possibly disparate invariant limit sets. We show that, under suitable conditions, trajectories globally converge to a superset of the limit sets and then remain in a second, larger superset. We show the effectiveness of the dwell-time conditions by using examples of switching limit cycles commonly found in robotic locomotion and flapping flight.

*Keywords:* Switched systems, Non-equilibrium steady state, Set-based control

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## 1. Introduction

Bifurcations have been of interest to dynamical systems theory for decades. However, most control strategies view such behavior as damaging and try to mitigate it [1]. Relatively less work actively inserts bifurcations as part of a control strategy. One example is using a classic Hopf bifurcation for mode-switching between flapping and gliding flight in micro-aerial vehicles [2]. The authors consider a supercritical Andronov-Hopf bifurcation model of  $\mathbf{x} = (u; v)$ :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \rho) = \begin{pmatrix} -\lambda/\rho^2 (u^2 + v^2 - \rho^2\sigma) u - \omega(t)v \\ \omega(t)u - \lambda/\rho^2 (u^2 + v^2 - \rho^2\sigma) v \end{pmatrix} \quad (1)$$

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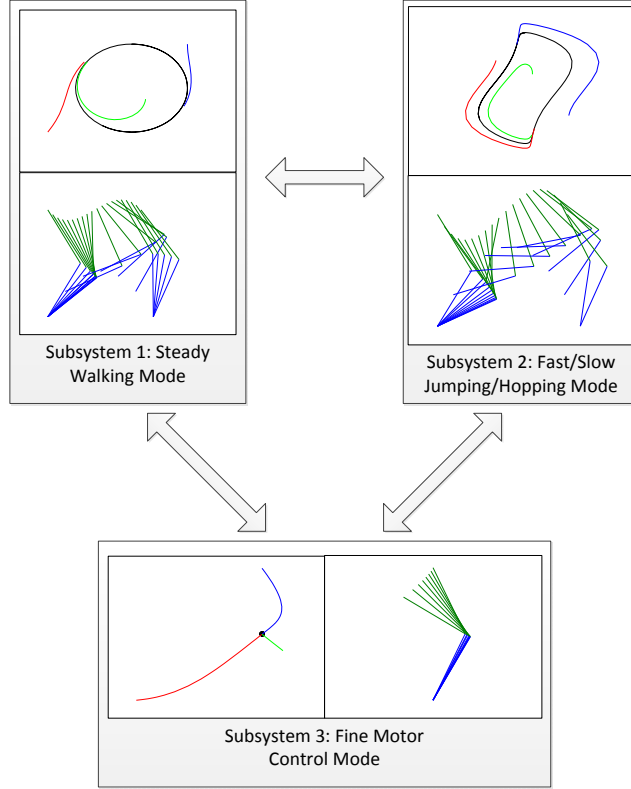


Figure 1: Schematic of mode switching with non-equilibrium limit sets.

with  $\sigma = 1$ . For a positive rate of convergence  $\lambda > 0$ , it can be easily shown that any initial trajectory  $(u; v) \neq \mathbf{0}$  exponentially converges to a circle of the radius  $\rho$  rotating at the time-varying frequency  $\omega(t)$  with bounded  $\dot{\omega}(t)$ . If  $\sigma \leq 0$ , bifurcation occurs and the system globally converges to the origin, which is useful for fast inhibition of oscillation. Fast inhibition and synchronization of oscillators are key properties for many neurobiologically-inspired control schemes.

Another possible application is walking robots. Fig. 1 shows a hypothetical switching pattern for a walking robot application utilizing central pattern generation. A guidance/navigation engineer may design limit cycle subsystems for walking and jumping modes (shown as a Hopf oscillator and a Van der Pol oscillator), while utilizing steady-state control strategies for static balancing or tasks requiring fine motor control.

Mode-switching also implicates a large body of literature on switched

systems [3]. Most work on stability of switched systems assumes that all subsystems have a common equilibrium point. [4, 5, 6] consider weak Lyapunov functions in the style of LaSalle for a common equilibrium. [7] considers equilibrium location changes, but holds the vector field constant. They connect the result to averaging theory. [8] considers practical stability of affine systems with multiple distinct equilibria. Alpcan and Başar investigated dwell time criteria for nonlinear globally exponentially stable subsystems which could have differing equilibria [9]. Such systems have no single globally attractive equilibrium point. The authors of [9] reported an explicit construction of the dwell time and a conservative invariant set. This paper is inspired by that work and is a generalization of it. We generalize their result to switched systems where each subsystem may have multiple invariant sets. We pursue a similar dwell time strategy in order to provide spatial bounds for the switched system. The resulting construction is slightly more complicated, as we consider  $\dot{V}$  in order to isolate the invariant sets rather than using the Lyapunov function alone.

Systems with bifurcation often contain multiple  $\omega$ -limit sets which cannot be globally exponentially stable. Instead, results such as LaSalle's invariant set theorem [10] allow us to analyze asymptotic stability of this larger class of systems. LaSalle's theorem and much of the switched systems literature are both Lyapunov-based, and we will make use of Lyapunov functions to define all the relevant sets. The benefit to relying on Lyapunov functions is that this requires no special structure on the subsystems' entire vector fields. The tradeoff is that we fail to exploit any special structure the subsystems may possess and the result relies on being able to find suitable Lyapunov functions.

Section 2 provides background assumptions and definitions. Section 3 begins by reconsidering existing results. Sections 3.3 through 3.5 present two methods to accomplish the goal. Choice of a particular method will depend on specific situations and design constraints. Section 4 shows a numerical example, and Section 5 provides closing comments.

## 2. Preliminaries and Definitions

Consider a set of continuous-time dynamical systems defined by

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad (2)$$

where  $\mathbf{x} \in \mathbb{R}^n$  and  $p \in \mathcal{P}$ , with some index set  $\mathcal{P} = \{p_1, p_2, \dots, p_{\max}\}$ . A piecewise constant switching signal  $\sigma : [0, \infty) \rightarrow \mathcal{P}$  specifies the active subsystem at each time. Assume, for ease of analysis, that  $\mathbf{f}_p$  are each continuous with continuous first partials. Together, (2), the index set, and the switching signal define a switched system.

Only some systems admit stability results for arbitrary switching signals, so we will consider a constraint on how quickly the switching signal can make consecutive switches.

**Definition** Consider a switched system with switching times  $\{t_1, t_2, \dots\}$ . It is said to have *dwell time*  $\tau$  if  $t_{i+1} - t_i \geq \tau \ \forall i \in \mathbb{N}$ .

Next, we review and introduce some important subsets of  $\mathbb{R}^n$ . We have not yet provided strict assumptions on Lyapunov-like functions. At present, it is enough to assume that each subsystem has a (possibly different)  $\mathcal{C}^1$  Lyapunov-like function, which is bounded above and below on every bounded subset of  $\mathbb{R}^n$ . Furthermore, assume that each is radially unbounded ( $V_p(\mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{x}\| \rightarrow \infty$ ). This ensures that every sublevel set describes a compact region. We assume for the remainder of this paper that the minimum value of each  $V_p$  is zero. Define

$$\mathcal{G}_p = \{\mathbf{x} \in \mathbb{R}^n | V_p(\mathbf{x}) = 0\} \quad (3)$$

as the set which attains the minimum value of  $V_p$ . Let  $\kappa$  be a positive constant and define

$$\mathcal{N}_p(\kappa) = \{\mathbf{x} \in \mathbb{R}^n | V_p(\mathbf{x}) \leq \kappa\}, \quad (4)$$

a closed  $\kappa$ -neighborhood of  $\mathcal{G}_p$ . For the purposes of Theorem 1,  $\mathcal{N}_p(\kappa)$  is connected, but it is not necessarily connected in the remainder of the paper (See Figure 4). Let the union over  $\mathcal{P}$  be

$$\mathcal{N}(\kappa) = \bigcup_{p \in \mathcal{P}} \mathcal{N}_p(\kappa). \quad (5)$$

Additionally, we define a superset,  $\mathcal{M}(\kappa)$ , in a series of steps with

$$\alpha_p(\kappa) = \max_{\mathbf{x} \in \mathcal{N}(\kappa)} V_p(\mathbf{x}), \quad (6)$$

and

$$\mathcal{M}_p(\kappa) = \{\mathbf{x} \in \mathbb{R}^n : V_p(\mathbf{x}) \leq \alpha_p(\kappa)\}. \quad (7)$$

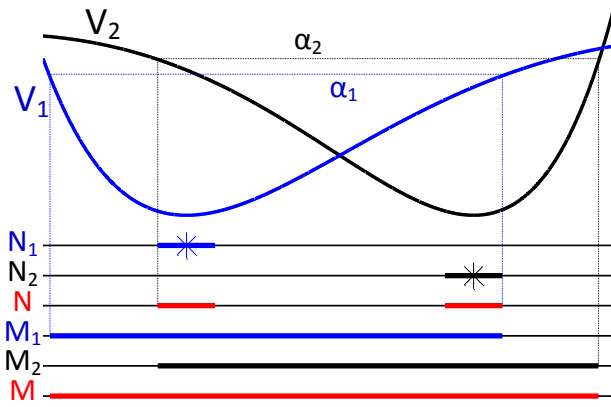


Figure 2: Qualitative example of how  $\mathcal{N}$  and  $\mathcal{M}$  are built for a switched system consisting of two subsystems, each with a single equilibrium, but at different locations.

Finally, we create a closed union of closed sublevel sets,

$$\mathcal{M}(\kappa) = \bigcup_{p \in \mathcal{P}} \mathcal{M}_p. \quad (8)$$

Notice that the dependence on  $\kappa$  carries through once we use it in  $\mathcal{N}(\kappa)$ . For the purposes of Theorem 1,  $\mathcal{M}$  is a connected superset of  $\mathcal{N}$ . Theorem 4 will introduce a different notion which is not necessarily connected. Figure 2 provides a one-dimensional example to help visualize these sets.

### 3. Stability Results

We first restructure the result in [9] slightly to add clarity and to better facilitate the generalization presented in this paper.

#### 3.1. Unique Equilibrium Case

**Theorem 1** ([9]). *Consider a family of systems defined by (2), each with a single, globally exponentially stable equilibrium, denoted  $\mathbf{x}_p^*$ . Suppose that the exponential decay rate of each Lyapunov function, as described in*

$$\dot{V}_p(\mathbf{x}) \leq -\varepsilon V_p(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n, \forall p \in \mathcal{P}, \quad (9)$$

*is at least  $\varepsilon > 0$ .*

*Furthermore, given a positive constant  $\kappa$ , define the sets as in Section 2 and assume  $\mu(\kappa) \in (1, \infty)$  such that*

$$\frac{V_r(\mathbf{x})}{V_q(\mathbf{x})} \leq \mu(\kappa), \forall q, r \in \mathcal{P} \quad \forall \mathbf{x} \in \mathbb{R}^n \setminus \mathcal{N}(\kappa). \quad (10)$$

Then, for every switching signal with dwell time

$$\tau > \frac{\log \mu(\kappa)}{\varepsilon}, \quad (11)$$

- (i) There exists a time  $T$  such that  $\mathbf{x}(T^-) \in \mathcal{N}_{\sigma(T^-)}(\kappa)$ , and
- (ii) For any time  $\bar{t}$  such that  $\mathbf{x}(\bar{t}) \in \mathcal{N}_{\sigma(\bar{t})}(\kappa)$ ,  $\mathbf{x}(t) \in \mathcal{M}(\kappa)$  for all  $t \geq \bar{t}$ .

**Remark 1.** Not all choices of  $V_p$  will give a finite value of  $\mu(\kappa)$ . Typical polynomial constructions for Lyapunov functions must have the same polynomial order. Scaling or stretching Lyapunov functions may be useful, but there will be implications for both the spatial parameter  $\mu$  and the temporal parameter  $\varepsilon$ . We will revisit the idea of scaling briefly in Sections 3.3-3.4.

*Proof.* We will only provide a sketch of the relevant features. The proof proceeds in two parts:

- (i) Consider a finite time interval  $[t_0, T]$  with corresponding switching times  $t_1 < t_2 < \dots < t_{n_\sigma}$ , where  $n_\sigma$  is the number of switches inside the interval. Between switches,  $\sigma(t)$  is constant. If the trajectory enters  $\mathcal{N}_{\sigma(t)}$  (not just  $\mathcal{N}$ ), the result is trivial. Otherwise, the behavior of  $V_{\sigma(t)}(\mathbf{x}(t))$  between switches satisfies (9). Denote the limit from the right/left as superscript  $+/-$ , respectively. Then,

$$V_{\sigma(t_{i+1}^-)}(\mathbf{x}(t_{i+1}^-)) \leq e^{-\varepsilon(t_{i+1}-t_i)} V_{\sigma(t_i^+)}(\mathbf{x}(t_i^+)). \quad (12)$$

At switches,

$$V_{\sigma(t_i^+)}(\mathbf{x}(t_i^+)) \leq \mu(\kappa) V_{\sigma(t_i^-)}(\mathbf{x}(t_i^-)) \quad (13)$$

holds. We can iterate on  $i$  to obtain

$$V_{\sigma(T^-)}(\mathbf{x}(T^-)) \leq e^{((\log \mu(\kappa)/\tau) - \varepsilon)(T-t_0)} V_{\sigma(t_0)}(\mathbf{x}(t_0)). \quad (14)$$

Importantly, under the dwell time condition, (11), this implies that by taking  $T$  suitably large, we can make  $V_{\sigma(T^-)}(\mathbf{x}(T^-))$  arbitrarily small. Thus,  $\mathbf{x}(T^-) \in \mathcal{N}_{\sigma(T^-)}(\kappa)$ . This proof is existential, not constructive. We cannot calculate a particular time  $T$  for any particular problem.

- (ii) The second part of the proof shows that after a switch at time  $t_i$ , the dwell time is sufficiently large to force the trajectory back into  $\mathcal{N}_{\sigma(t)}(\kappa)$  before a subsequent switch at time  $t_{i+1}$ . Furthermore, the trajectory cannot escape  $\mathcal{M}(\kappa)$  in that interval.

□

Details are available in [9]. The proof presented above is different from [9] in one important way - we specify that the trajectory enters  $\mathcal{N}_{\sigma(t)}(\kappa)$  rather than  $\mathcal{N}(\kappa)$ . In fact, it is an error to do the latter. The trajectory may pass through  $\mathcal{N}(\kappa) \setminus \mathcal{N}_{\sigma(t)}(\kappa)$  and then switch after it has emerged, which may cause it to exit  $\mathcal{M}(\kappa)$ . There is nothing special about  $\mathcal{N}(\kappa) \setminus \mathcal{N}_{\sigma(t)}(\kappa)$  while  $\sigma(t)$  remains constant. It may be aesthetically displeasing to have the entry set change in time along with  $\sigma(t)$ , but we must do this, because the time-independent formulation is false. We will demonstrate with an example.

**Example** This example is a slight variation of Example 2 from [9]. Choose

$$\dot{\mathbf{x}}_p = A\mathbf{x}_p + b_p, \quad (15)$$

but with

$$A = \begin{bmatrix} -1 & -10 \\ 10 & -1 \end{bmatrix} \quad (16)$$

$$b_1 = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ 10 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ -10 \end{bmatrix}.$$

We are able to use the same Lyapunov functions,  $V_1(\mathbf{x}) := x_1^2 + (x_2 - 1)^2$ ,  $V_2(\mathbf{x}) := (x_1 + 1)^2 + x_2^2$ , and  $V_3(\mathbf{x}) := (x_1 - 1)^2 + x_2^2$ . One can check that  $\tau$  can be the same as in [9], but that is not important here. Consider the trajectory shown in Figure 3. The red circles show  $\mathcal{N}(\kappa)$ ; the black circles show  $\mathcal{M}(\kappa)$ . Subsystem 1 is active at the start, and the trajectory is shown in blue.  $\mathcal{N}_1(\kappa)$  is the red circle centered around  $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ . The trajectory passes through  $\mathcal{N}_3(\kappa)$ , which is the red circle centered around  $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ . This is entering  $\mathcal{N}(\kappa)$ , but not  $\mathcal{N}_{\sigma(t)}(\kappa)$ . Now, notice that we could have selected initial conditions to make the entry into  $\mathcal{N}_3(\kappa)$  occur at arbitrarily large time. This allows us to place a switch anywhere along the trajectory, regardless of  $\tau$  (this is a “free switch” that we will see again in Section 3.5). In this case, we switched to subsystem 2 at the location where the trajectory changes to green. It exits  $\mathcal{M}(\kappa)$  shortly thereafter.

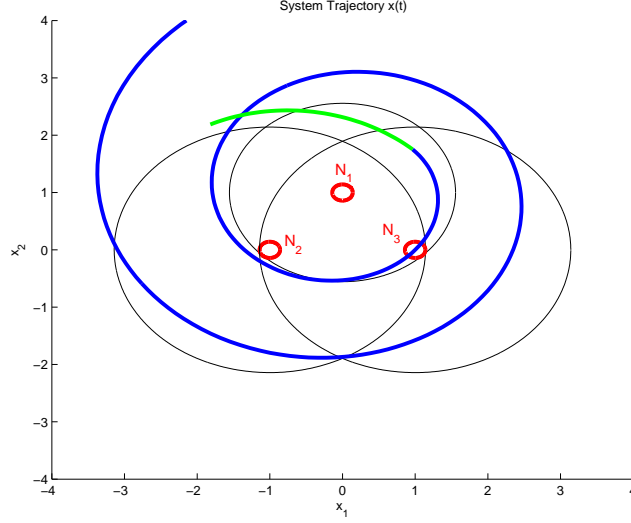


Figure 3: Example trajectory demonstrating the need for time-dependent  $\mathcal{N}_{\sigma(t)}(\kappa)$ .

### 3.2. Problem Statement: Switching Systems having Multiple Invariant Sets

Alpcan and Başar considered subsystems, each having a globally exponentially stable equilibrium point [9]. We will relax this condition to allow for systems with multiple invariant sets. Consider a switched system with  $C^1$  functions  $V_p : \mathbb{R}^n \rightarrow \mathbb{R}$  bounded on every bounded subset of  $\mathbb{R}^n$  such that

$$\dot{V}_p(\mathbf{x}) \leq 0, \forall \mathbf{x} \in \mathbb{R}^n, \quad (17)$$

and  $V_p(\mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{x}\| \rightarrow \infty$ . Denote

$$E_p = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \dot{V}(\mathbf{x}) = 0 \right\}. \quad (18)$$

Our problem is as follows. Is there a dwell time condition that suffices for ultimate boundedness?

Two challenges are immediately apparent. First, if  $\dot{V}$  vanishes outside of  $\mathcal{N}_p(\kappa)$ , there is not a strictly positive Lyapunov decay rate outside of  $\mathcal{N}_p(\kappa)$ , so it is unclear how long to wait between switches. Second, even in the absence of switching, the active system may never enter  $\mathcal{N}_p(\kappa)$  ( $E_p$  may not be contained in  $\mathcal{N}_p(\kappa)$ ; see Figure 4). The following three subsections describe two methods for overcoming these challenges.

### 3.3. Intermediate Solution: Expand Entry Neighborhood

The simplest idea is to realize that  $\mathcal{N}_p(\kappa)$  grows in size as we increase  $\kappa$ . Assuming that  $E_p$  is bounded, we can pick  $\kappa$  large enough so that  $E_p \subset \mathcal{N}_p(\kappa)$ . Even so, not all Lyapunov functions satisfying (17) will decay exponentially on  $\mathbb{R}^n \setminus \mathcal{N}_p(\kappa)$ . For example, consider a single one-dimensional subsystem  $\dot{x} = -\arctan(x)$  with  $V = x^2$ . Nevertheless, the following lemma is useful:

**Lemma 2.** *Consider a switched system with Lyapunov functions  $\mathcal{V}_p$  satisfying  $\dot{\mathcal{V}}_p \leq 0$  with  $\mathcal{V}_p(\mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{x}\| \rightarrow \infty$  and  $E_p$  bounded. Then, for sublevel Lyapunov sets  $\mathcal{N}_p(\kappa)$  such that  $E_p \subset \text{Int}(\mathcal{N}_p(\kappa))$  (where  $\text{Int}$  denotes the interior) and any  $\varepsilon \in \mathcal{R}$ , there exists a different set of continuous, radially unbounded Lyapunov functions  $V_p$  such that,*

1.  $V_p(\mathbf{x}) = 0$  for  $\mathbf{x} \in \mathcal{N}_p$ , and  $V_p(\mathbf{x}) > 0$  for  $\mathbf{x} \in \mathbb{R}^n \setminus \mathcal{N}_p$ .
2.  $V_p(\mathbf{x}(t + t_0)) \leq e^{-\varepsilon t} V_p(\mathbf{x}(t_0))$  for all  $\mathbf{x} \in \mathbb{R}^n \setminus \mathcal{N}_p$  and  $t, t_0 \in \mathbb{R}$ .

*Proof.* Bhatia [11] constructed a unique continuous function  $s(\mathbf{x})$  on  $\mathbb{R}^n \setminus \mathcal{N}_p$  such that  $s(\mathbf{x}(t + t_0)) = s(\mathbf{x}(t_0)) - t$  and  $s(\mathbf{x}) \rightarrow 0$  as  $\mathbf{x} \rightarrow \mathcal{N}_p$ . Then, we can select any constant  $\varepsilon$  and set

$$V_p(\mathbf{x}) = \begin{cases} 0 & \text{for } \mathbf{x} \in \mathcal{N}_p \\ e^{\varepsilon s(\mathbf{x})} & \text{for } \mathbf{x} \in \mathbb{R}^n \setminus \mathcal{N}_p. \end{cases} \quad (19)$$

Hence, on  $\mathbb{R}^n \setminus \mathcal{N}_p$ ,

$$V_p(\mathbf{x}(t + t_0)) = e^{\varepsilon s(\mathbf{x}(t+t_0))} = e^{\varepsilon(s(\mathbf{x}(t_0)) - t)} = e^{-\varepsilon t} V_p(\mathbf{x}(t_0)). \quad (20)$$

□

This means that for suitably large  $\kappa$ , there exist  $V_p$  that decay exponentially outside  $\mathcal{N}_p(\kappa)$ . A simple constant shift can patch the original Lyapunov function on  $\mathcal{N}_p(\kappa)$  with the construction of Lemma 2 outside  $\mathcal{N}_p(\kappa)$  at  $\partial\mathcal{N}_p$  (the boundary of  $\mathcal{N}_p$ ), and the mere continuity of  $V_p$  outside of  $\mathcal{N}_p$  does not harm any essential parts of the proof. (Note that performing a constant shift outside  $\mathcal{N}_p(\kappa)$  will scale  $\varepsilon$  by  $\alpha_p$ , but we could just perform the construction again with a larger  $\varepsilon$  to correct for this). However, relying on the construction of Lemma 2 may not allow for (10) to hold, and we must assume that we can find a set of Lyapunov functions which satisfy both the exponential decay property and the  $\mu$  property, (10). It is left as an open problem to

determine if it is generally possible to construct a set of Lyapunov functions that satisfy both conditions for any switched system.

Now, the conditions required for Theorem 1 hold (noting that is is not essential to have exponential Lyapunov decay within  $\mathcal{N}_p$ ).

**Remark 2.** It is useful to notice that our set definitions do not require a single value of  $\kappa$  for all subsystems. If our subsystems have qualitatively different invariant sets, it may be very damaging to require a single  $\kappa$ . Instead, we can choose a set,  $\boldsymbol{\kappa} = \{\kappa_p\}$ , of values and restate all our set definitions as functions of  $\boldsymbol{\kappa}$ .

Making  $\kappa_p$  very large is problematic in two ways. First, the purpose of defining a large neighborhood is to neglect the troubling areas (by this, we mean that we do not know much about the trajectory inside  $\mathcal{N}(\boldsymbol{\kappa})$ ). However, with large  $\kappa_p$ , we may be cutting out substantial portions of our state space. Secondly, it may lead to larger dwell time or larger  $\mathcal{M}(\boldsymbol{\kappa})$ . In the next two subsections, we describe a method to more tightly tailor our strategy.

### 3.4. First Main Result: Tightly Tailored Entry Set

Since the troubling regions are just those where  $\dot{V}_p \leq 0$  is closest to zero, we will now define a smaller set containing these regions. Choose a set  $\boldsymbol{\delta} = \{\delta_{p_1}, \delta_{p_2}, \dots, \delta_{p_{\max}}\}$  with  $\delta_{p_i} > 0$ , and define the set

$$\mathcal{H}_p(\delta_p) = \left\{ \mathbf{x} \in \mathbb{R}^n : \dot{V}_p(\mathbf{x}) > -\delta_p \right\}, \quad (21)$$

so that  $E_p \subset \mathcal{H}_p$ . As usual, we can also define  $\mathcal{H}(\boldsymbol{\delta}) = \bigcup_{p \in \mathcal{P}} \mathcal{H}_p$ . Figure 4 provides a one-dimensional example to help visualize how these sets are constructed. Similar to before, not all  $V_p$  satisfying (17) decay exponentially outside  $\mathcal{H}_p(\delta_p)$ . Define

$$\gamma_p(\delta_p) = \max_{\mathbf{x} \in \mathcal{H}_p} V_p(\mathbf{x}). \quad (22)$$

Furthermore, set

$$\mathcal{L}_p(\delta_p) = \left\{ \mathbf{x} \in \mathbb{R}^n : V_p(\mathbf{x}) \leq \gamma_p(\delta_p) \right\}. \quad (23)$$

Since  $\mathcal{L}_p$  is a sublevel Lyapunov set, it is compact. From compactness,  $\dot{V}_p$  attains a minimum on  $\mathcal{L}_p$ , while  $V_p$  attains a finite maximum. Thus, on  $\mathcal{L}_p \setminus \mathcal{H}_p$ , an exponential decay rate can be computed, while we can use Lemma 2



*Proof.* Notice that all of the necessary assumptions are valid on  $\mathbb{R}^n \setminus \mathcal{H}_{\sigma(t)}(\boldsymbol{\delta})$ . By taking suitably large  $T$ , we can make  $V_{\sigma(T^-)}(\mathbf{x}(T^-))$  arbitrarily small. Therefore, either  $\mathbf{x}(T^-) \in \mathcal{N}_{\sigma(T^-)}(\boldsymbol{\kappa}) \subseteq \mathcal{H}_{\sigma(T^-)}(\boldsymbol{\delta})$  or the trajectory enters  $\mathcal{H}_{\sigma(t)}(\boldsymbol{\delta})$  somewhere else before that time.  $\square$

**Remark 3.** In [9],  $\kappa$  was a single tuning parameter. In Section 3.3,  $\boldsymbol{\kappa} = \{\kappa_p\}$  was introduced as a possible set of tuning parameters. Now, per Theorem 3, the set of tuning parameters is  $\boldsymbol{\delta} = \{\delta_p\}$ , and  $\boldsymbol{\kappa}$  is computed as a consequence of our selection of  $\boldsymbol{\delta}$ . Since nearly every parameter/set which follows is dependent on  $\boldsymbol{\delta}$ , we will often omit explicit dependence in favor of readability.

### 3.5. Second Main Result: No-Escape Set

This section assumes the trajectory has entered  $\mathcal{H}_{\sigma(t)}(\boldsymbol{\delta})$  at some time and proceeds to build the relevant no-escape set, which will be denoted  $\mathcal{M}_p$ .

The primary problem is that  $\mathcal{H}_p$  is not necessarily an invariant set even for periods of time when  $\sigma(t)$  constant is constant (i.e., no switching). For example, a subsystem may contain a locally unstable equilibrium. With  $\delta_p$  small,  $\mathcal{H}$  is certainly not an invariant set. One way this can be problematic is that we can get a free extra switch.

**Example** Consider two one-dimensional subsystems,  $\dot{x}_1 = -x_1$  and  $\dot{x}_2 = x_2 - x_2^3$ , with  $V_2 = (x - 1)^2(x + 1)^2$  and small  $\delta_2$ . Assume a nonzero initial condition with the first subsystem being active. The trajectory can become arbitrarily close to zero before switching to the second subsystem. While the second subsystem is active ( $\sigma(t)$  is constant), the trajectory can clearly leave  $\mathcal{H}_2$ . Furthermore, since the second subsystem started arbitrarily close to the origin, it can take an arbitrarily long time to exit  $\mathcal{H}_2$ . Thus, no finite dwell time can prevent at least one switch from being possible outside of  $\mathcal{H}$ .

There are two ways to compute a dwell time and an associated spatial bound, but we need a few more definitions first. Set the usual  $\gamma(\boldsymbol{\delta}) = \max_{p \in \mathcal{P}} \gamma_p(\delta_p)$  and  $\mathcal{L}(\boldsymbol{\delta}) = \bigcup_{p \in \mathcal{P}} \mathcal{L}_p$ . Compute

$$\xi_p(\boldsymbol{\delta}) = \max_{\mathbf{x} \in \mathcal{L}(\boldsymbol{\delta})} V_p(\mathbf{x}), \quad (26)$$

and set  $\xi(\boldsymbol{\delta}) = \max_{p \in \mathcal{P}} \xi_p(\boldsymbol{\delta})$ .

**Theorem 4.** Consider a family of systems defined by (2), each with a radially unbounded Lyapunov-like function that satisfies (17). Assume  $\mathcal{H}_p$ ,  $E_p$  bounded and  $\mathcal{G}_p \subset \mathcal{H}_p(\delta)$ . Compute  $\varepsilon(\delta)$  so that (24) holds and assume

$$\frac{V_r(\mathbf{x})}{V_q(\mathbf{x})} \leq \mu(\delta), \forall q, r \in \mathcal{P} \quad \forall \mathbf{x} \in \mathbb{R}^n \setminus \mathcal{H}(\delta) \quad (27)$$

holds for finite  $\mu(\delta)$ . Furthermore, compute  $\kappa_p > 0$  such that  $\mathcal{G}_p \subset \mathcal{N}_p(\kappa) \subseteq \mathcal{H}_p(\delta)$ . Set  $\kappa = \min_{p \in \mathcal{P}} \kappa_p$ . Then, for every switching signal with dwell time

$$\tau > \frac{\log \frac{\xi(\delta)}{\kappa}}{\varepsilon}, \quad (28)$$

for every  $\bar{t}$  such that  $\mathbf{x} \in \mathcal{H}_{\sigma(\bar{t})}$ ,  $\mathbf{x}(t) \in \mathcal{M}(\delta) = \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$  for all  $t \geq \bar{t}$ , where

$$\mathcal{M}_p(\delta) = \{\mathbf{x} \in \mathbb{R}^n : V_p(\mathbf{x}) \leq \xi_p\}. \quad (29)$$

*Proof.* Consider the following sequence of times. Assume a switch occurs or the system is started at  $t_0$ . Assume further that there is a time  $t_0 \leq t_{\text{enter}}$  at which the trajectory enters  $\mathcal{H}_{\sigma(t_0^+)}(\delta)$ . There may or may not be a time at which the trajectory exits  $\mathcal{H}_{\sigma(t_0^+)}(\delta)$ , which we will denote  $t_{\text{enter}} \leq t_{\text{exit}}$ . If there is,  $t_{\text{exit}} - t_{\text{enter}}$  could possibly be arbitrarily large. Therefore, the second switching time  $t_1$  may be shortly after  $t_{\text{exit}}$  while the trajectory is outside of  $\mathcal{H}_{\sigma(t_0^+)}(\delta)$  regardless of the dwell time.

The proof will proceed in two parts. First, we will show that the trajectory does not leave  $\mathcal{M}$  before a third switching time, called  $t_2$ . Then, we will compute a dwell time so that the trajectory must re-enter  $\mathcal{H}_{\sigma(t)}$  before  $t_2$ . By induction, the trajectory will never leave  $\mathcal{M}$ .

- (i) We will show that the trajectory never leaves  $\mathcal{M}$  on  $[t_{\text{enter}}, t_1]$ , then on  $[t_1, t_2]$ . Since  $\mathbf{x}(t_{\text{enter}}) \in \mathcal{H}_{\sigma(t_0)} \subset \mathcal{M}_{\sigma(t_0)}$  and  $\mathcal{M}_{\sigma(t_0)}$  is a sublevel Lyapunov set, the trajectory cannot leave  $\mathcal{M}_{\sigma(t_0)}$  on  $[t_{\text{enter}}, t_1]$  since no switching occurs on this interval. Next, notice

$$\mathbf{x}(t_1) \in \mathcal{L}_{\sigma(t_1^-)} \subset \mathcal{L} \subset \mathcal{M}_{\sigma(t_1^+)}. \quad (30)$$

Thus, the trajectory cannot leave  $\mathcal{M}_{\sigma(t_1^+)}$  on  $[t_1, t_2]$ .

- (ii) We will show that the trajectory must re-enter  $\mathcal{H}_{\sigma(t)}$  before  $t_2$ . Define  $\Delta = t_2 - t_1 \geq \tau$ , giving

$$V_{\sigma(t_2^-)}(\mathbf{x}(t_2^-)) \leq e^{-\varepsilon\Delta} V_{\sigma(t_1^+)}(\mathbf{x}(t_1^+)) \leq e^{-\varepsilon\Delta} \xi_{\sigma(t_1^+)} \leq e^{-\varepsilon\Delta} \xi. \quad (31)$$

The dwell time given by (28) gives that  $V_{\sigma(t_2^-)}(\mathbf{x})(t_2^-) \leq \kappa$ . Thus, the trajectory enters  $\mathcal{N}_{\sigma(t_1^+)}$  or reenters  $\mathcal{H}_{\sigma(t)}$  elsewhere before time  $t_2$ .

□

**Corollary 5.** *Taken together, Theorems 3 and 4 provide a dwell time which guarantees finite time entry into  $\mathcal{H}_{\sigma(t)}$  and the invariance of  $\mathcal{M}$  thereafter. Given a particular problem, if we want to apply both theorems, we should set  $\tau > \frac{\log \max(\mu, \frac{\xi}{\kappa})}{\varepsilon}$ .*

Section 4 will show examples that have  $\frac{\xi}{\kappa} > \mu$ , but it is trivial to construct cases with very large  $\mu$ . Next, we construct some lower bounds for  $\mu$  which are helpful for determining that the bounds in these theorems are tighter than some other possible bounds. Define

$$\eta_p(\boldsymbol{\delta}) = \max_{\mathbf{x} \in \mathcal{H}} V_p(\mathbf{x}), \quad (32)$$

and the following proposition will be useful.

**Proposition 6.** *Given the preceding set definitions,  $\mu\gamma \geq \xi_p \geq \eta_p \ \forall p$ .*

*Proof.* Since  $\mathcal{H}_p \subset \mathcal{L}_p$ ,  $\mathcal{H} \subset \mathcal{L}$ , and  $\xi_p \geq \eta_p$ .

Without loss of generality, consider  $\mathbf{x} \in \mathcal{L}$  such that  $V_1(\mathbf{x}) = \xi_1$ . There are three cases:

**Case 1.** *Suppose  $\mathbf{x} \in \mathcal{H}_1$ . Then,  $\xi_1 = \gamma_1$ . Since  $\gamma \geq \gamma_1$  and  $\mu \geq 1$ ,  $\mu\gamma \geq \xi_1$ .*

**Case 2.** *Suppose  $\mathbf{x} \in \mathcal{H} \setminus \mathcal{H}_1$ . Then,  $\xi_1 = \eta_1$ . Since  $\xi_1$  is a maximum of  $V_1$  over  $\mathcal{L} \supset \mathcal{H}$  and  $\mathcal{H}$  is open,  $V_1(\mathbf{x})$  must be a local maximum. But, if it is a local maximum,  $\nabla V_1(\mathbf{x}) = 0$  and  $\dot{V}_1(\mathbf{x}) = 0$ . Thus,  $\mathbf{x} \in \mathcal{H}_1$ , and the problem is reduced to Case 1.*

**Case 3.** *Suppose  $\mathbf{x} \in \mathcal{L} \setminus \mathcal{H}$ . Then, (27) holds, and*

$$\mu \geq \frac{V_1(\mathbf{x})}{V_p(\mathbf{x})} \ \forall p. \quad (33)$$

*Then, for some  $p$ ,  $\mu\gamma \geq \mu\gamma_p \geq \mu V_p(\mathbf{x}) \geq V_1(\mathbf{x}) = \eta_1 = \xi_1$ .*

□

Notice that in the ideal case, with each subsystem consisting only of a single exponentially stable equilibrium,  $\gamma = \kappa$ ,  $\mathcal{L}_p = \mathcal{H}_p = \mathcal{N}_p$ , and  $\xi = \alpha$  can be attained. In the case they are attained,  $\tau = \frac{\log \max(\mu, \frac{\xi}{\kappa})}{\varepsilon} = \frac{\log \max(\mu, \frac{\xi}{\gamma})}{\varepsilon} = \frac{\log \mu}{\varepsilon}$ , (29) collapses to (7), and we can consider this a true generalization of the result in [9]. In fact, one can notice that both examples in [9] satisfy the equality  $\mu\kappa = \xi$ .

**Remark 4.** Using similar techniques as those in Theorem 4, one can find a bound for  $\tau$  and  $\mathcal{M}$  that looks like  $\min(\max(\mu\gamma, \eta_p), \xi_p)$ . This expression reduces to  $\xi_p$  given Proposition 6.

**Remark 5.** We chose to define the dwell time as a constant across all subsystems, but allowed the spatial calculation to proceed with regard to individual subsystems. This is consistent with the idea that applications will utilize a single, subsystem invariant, dwell time. Since only the final union matters for the no-escape set, it is of no importance that the individual pieces were built with different constants determining sublevel sets.

## 4. Examples

We provide two numerical examples to illustrate the main results of the paper. In particular, we want to demonstrate that tight tailoring can provide better results than expanding the entry neighborhood.

### 4.1. One-Dimensional Example

Consider a family of systems defined by

$$f_1(x) = -x(x-1)^2, \quad f_2(x) = -(x-1). \quad (34)$$

Select  $V_1(x) = x^2$  and  $V_2(x) = (x-1)^2$  and  $\delta = [0.05; 0.1]$ . Then,  $\mathcal{H}_1 = [-0.139, 0.197] \cup [0.803, 1.139]$ ,  $\mathcal{H}_2 = [0.776, 1.224]$ ,  $\mu = 67.28$ , and  $\varepsilon = 1.29$ . We can set  $\kappa = [0.0193; 0.050]$ , giving  $\mathcal{N}_1 = [-0.139, 0.139] \subset \mathcal{H}_1$  and  $\mathcal{N}_2 = [0.776, 1.224] \subset \mathcal{H}_2$ . Then, we can compute  $\gamma = [1.297; 0.050]$ ,  $\mathcal{L}_1 = [-1.139, 1.139]$ ,  $\mathcal{L}_2 = [0.776, 1.224]$ , and  $\xi = [1.497; 4.575]$ . Finally, this results in  $\tau = 4.240$  and  $\mathcal{M} = [-1.224, 3.139]$ . The size of  $\mathcal{M}$  is 4.362.

We compare the large- $\kappa$  method with two metrics. First, we select  $\kappa$  so that  $\tau$  is the same as the tight-tailoring method (we make the methods temporally-equivalent) and compare the size of  $\mathcal{M}$ . Second, we select  $\kappa$  so that the size of  $\mathcal{M}$  is the same as the tight-tailoring method (we make the

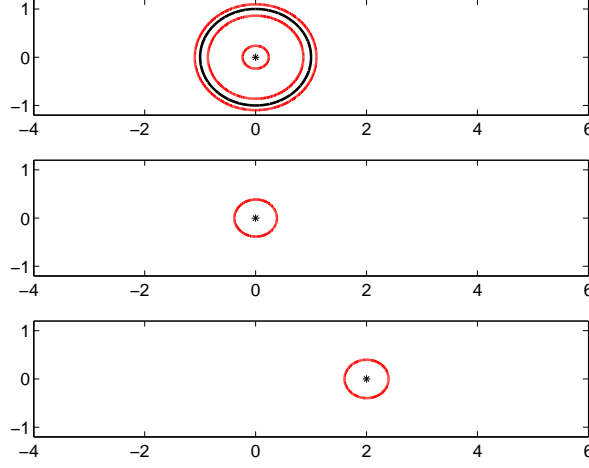


Figure 5: The sets  $E$  (black) and  $\mathcal{H}$  (red) for each subsystem.

methods spatially-equivalent) and compare the required dwell time. In order to make  $\tau = 4.240$ , we must set  $\kappa = [2.271; 0.257]$ . This results in  $\mu = 8.837$ ,  $\varepsilon = 0.514$ , and  $\mathcal{M} = [-1.507, 3.507]$ , which has size 5.014. For the second case, we can make  $\mathcal{M} = [-1.181, 3.181]$  by setting  $\kappa = [1.395; 0.0328]$ . The result is that  $\mu = 42.48$ ,  $\varepsilon = 0.0657$ , and  $\tau = 57.08$ . In each case, tight tailoring shows better performance.

#### 4.2. Two-Dimensional Limit Cycle Example

Consider a family of systems defined by

$$\begin{aligned} \mathbf{f}_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} -(x^2 + y^2 - 1) & -1 \\ 1 & -(x^2 + y^2 - 1) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{f}_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} -(x^2 + y^2 + 1) & -1 \\ 1 & -(x^2 + y^2 + 1) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{f}_3 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}. \end{aligned} \quad (35)$$

This is a simplified version of (1), a central pattern generator used for control of swimming [12] or flying robots [2]. Subsystem 1 has an unstable equilibrium at the origin and a stable limit cycle of unit radius centered around the origin. Subsystem 2 has a stable equilibrium at the origin. Subsystem 3 has a stable equilibrium at  $\begin{bmatrix} 2 & 0 \end{bmatrix}^T$ . Take the corresponding Lyapunov functions

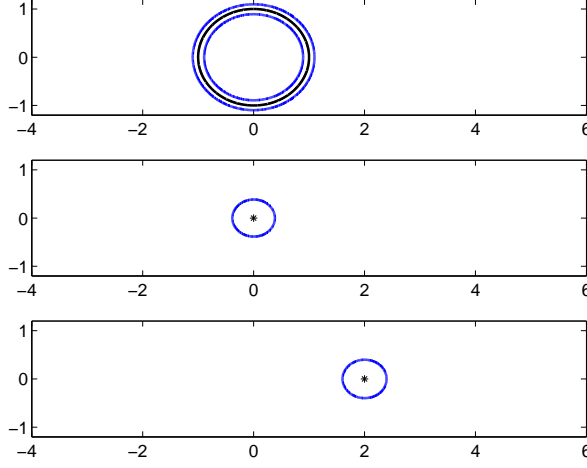


Figure 6: The sets  $\mathcal{G}$  (black) and  $\mathcal{N}$  (blue) for each subsystem. The example is suitably well-behaved so that  $E = \mathcal{G}$  and  $\mathcal{H} = \mathcal{N}$  except for the unstable origin and its surrounding neighborhood when  $p = 1$ .

to be  $V_1(\mathbf{x}) = (x^2 + y^2 - 1)^2$ ,  $V_2(\mathbf{x}) = (x^2 + y^2)^2$ , and  $V_3(\mathbf{x}) = ((x - 2)^2 + y^2)^2$ , respectively. If we choose  $\boldsymbol{\delta} = [0.2, 0.1, 0.1]$ , then  $\epsilon = 0.22$ ,  $\mu = 2215$ ,  $\boldsymbol{\kappa} = [0.042, 0.022, 0.025]$ ,  $\boldsymbol{\gamma} = [1.0, 0.22, 0.25]$ ,  $\boldsymbol{\xi} = [23, 33, 136]$ , and  $\tau = 38.9$ . Figures 5, 6, and 7 show the resulting sets. In fact, many of the resulting parameters (chief among them dwell time and size of  $\mathcal{M}$ ) are determined through nontrivial relations. Optimization and tradeoffs between spatial performance and temporal performance is problem-specific.

## 5. Conclusion

We derived a stability result for switched systems which are constructed from subsystems which possibly contain multiple invariant sets. This amounted to a generalization and refinement of the argument presented in [9] and is in the spirit of dwell time methods for switched systems. This result can be applied to a larger class of dynamical systems than those in [9], including those which contain bifurcations. We provide a set of tuning parameters,  $\boldsymbol{\delta}$ , which cause interrelated and problem-specific effects on performance.

Future studies could be focused on determining an explicit relation between tuning parameters and performance for a subclass of systems in order to better facilitate optimization or investigating a continuous family of switched systems near a bifurcation point.

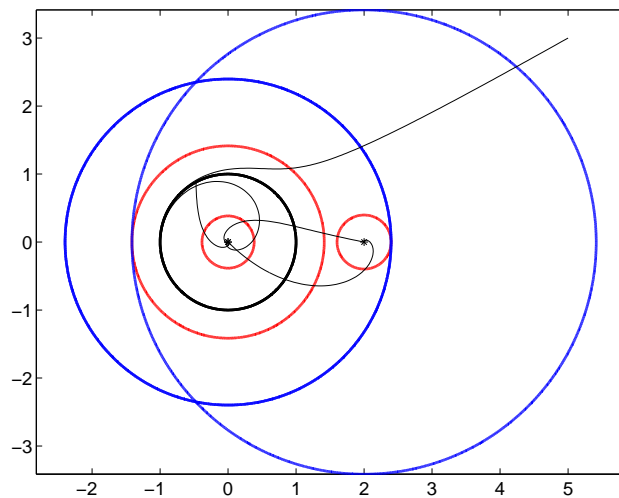


Figure 7: The sets  $E_p$  (black),  $\mathcal{L}$  (red), and  $\mathcal{M}$  (blue). The skinny black line is an example trajectory.

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